Multiplicity of the Interval Module

Contributors: Laurianne Baril and Justin Desrochers

Samuel Leblanc
samuel.leblanc6@usherbrooke.ca
Université de Sherbrooke, QC, Canada

Supervisor: Thomas Brüstle

$\mathcal{P}$-Persistence Modules

Let K be a field and let $\mathcal{P}$ be a finite and connected poset. We see $\mathcal{P}$ as the category with $\operatorname{Obj}\mathcal{P} = \mathcal{P}$ and with

$$\operatorname{Hom}_{\mathcal{P}}(x,y) = \begin{cases} \{x \to y\} & \text{if } x \le y, \\ \emptyset & \text{otherwise.} \end{cases}$$

Hence, if $\mathcal{S}$ and $\mathcal{P}$ are posets, then a poset homomorphism is simply a functor $F : \mathcal{S} \to \mathcal{P}$ and an order embedding ($x \le y \iff F(x) \le F(y)$) is a fully faithful functor. A connected and convex poset is said to be an *interval.*

Definition: A *$\mathcal{P}$-persistence module* is a functor $M : \mathcal{P} \to \operatorname{vect}_K$. We denote by $\operatorname{mod}\mathcal{P}$ the category of (pointwise finite-dimensional) $\mathcal{P}$-persistence modules.

Example: Let $\mathcal{S} \subseteq \mathcal{P}$ be an interval. The *interval module* for $\mathcal{S}$, denoted $\mathbb{I}_{\mathcal{S}} \in \operatorname{mod}\mathcal{P}$, is the functor

$$\mathbb{I}_{\mathcal{S}}(x) = \begin{cases} K & \text{if } x \in \mathcal{S}, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \mathbb{I}_{\mathcal{S}}(x \to y) = \begin{cases} \mathrm{id} & \text{if } x, y \in \mathcal{S}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $M, N \in \operatorname{mod}\mathcal{P}$. The *direct sum* of M and N, denoted $M \oplus N$, is the $\mathcal{P}$-persistence module with $(M \oplus N)(x) = M(x) \oplus N(x)$ and $(M \oplus N)(x \le y) = M(x \le y) \oplus N(x \le y)$ for all $x \le y \in \mathcal{P}$. If $M \cong M_1 \oplus M_2$ implies that $M_1 = 0$ or $M_2 = 0$, we say that M is *indecomposable.*

Example: If $\mathcal{P}$ is a linear order, then $M \in \operatorname{mod}\mathcal{P}$ is indecomposable if and only if $M \cong \mathbb{I}_{\mathcal{S}}$ for an interval $\mathcal{S} \subseteq \mathcal{P}$.

Krull-Schmidt Theorem ([HGK07], 3.1.1)

Let $M \in \operatorname{mod}\mathcal{P}$. Then, $M \cong \oplus_{i=1}^m M_i$ where M_i is indecomposable and this decomposition is unique up to isomorphism and order of summands.

Generalized Rank

We write $\lim M$ and $\operatorname{colim} M$ for the limit and the colimit of $M \in \operatorname{mod}\mathcal{P}$, respectively. Suppose $(\lambda_x)_{x\in\mathcal{P}} : \lim M \to M$ and $(\gamma_x)_{x\in\mathcal{P}} : M \to \operatorname{colim} M$ are the universal cones. There is a canonical map $\Psi_M : \lim M \to \operatorname{colim} M$ given by $\gamma_x \circ \lambda_x$, for any $x \in \mathcal{P}$. The fact that $\mathcal{P}$ is connected ensures that Ψ_M is well defined, i.e., that the definition does not depend on the choice of x.

Definition ([KM21]): The *generalized rank* of $M \in \operatorname{mod}\mathcal{P}$ is $\operatorname{rk} M \coloneqq \operatorname{rk}\Psi_M$.

Example: Let $\mathcal{P} = \{0,1,2\}^2$ and suppose $M \in \operatorname{mod}\mathcal{P}$ is

$$\begin{tikzcd}
M(0,2) \arrow[r, "M_{02}^{12}"] & M(1,2) \arrow[r, "M_{12}^{22}"] & M(2,2) \\
M(0,1) \arrow[u, "M_{01}^{02}"] \arrow[r, "M_{01}^{11}"] & M(1,1) \arrow[u, "M_{11}^{12}"] \arrow[r, "M_{11}^{21}"] & M(2,1) \arrow[u, "M_{21}^{22}"] \\
M(0,0) \arrow[u, "M_{00}^{01}"] \arrow[r, "M_{00}^{10}"] & M(1,0) \arrow[u, "M_{10}^{11}"] \arrow[r, "M_{10}^{20}"] & M(2,0) \arrow[u, "M_{20}^{21}"]
\end{tikzcd}$$

where $M_{ij}^{kl} = M((i,j) \to (k,l))$. Pick any path $(0,0) \rightsquigarrow (2,2)$, for instance the green one. Then, $\lim M = M(0,0)$, $\operatorname{colim} M = M(2,2)$, and $\Psi_M = M_{21}^{22}M_{11}^{21}M_{10}^{11}M_{00}^{10}$. More precisely, if $x = (1,0)$, then $\lambda_x = M_{00}^{10}$ and $\gamma_x = M_{21}^{22}M_{11}^{21}M_{10}^{11}$.

Theorem 1 ([CL19], 3.1)

Every $M \in \operatorname{mod}\mathcal{P}$ decomposes as $M \cong (\mathbb{I}_{\mathcal{P}})^{\operatorname{rk} M} \oplus N$ where $\operatorname{im}\Psi_N = 0$.

Proof: First, we define $(\operatorname{im}\Psi_M)_{\mathcal{P}}$ to be the $\mathcal{P}$-persistence module with $(\operatorname{im}\Psi_M)_{\mathcal{P}}(x) = \operatorname{im}\Psi_M$ and $(\operatorname{im}\Psi_M)_{\mathcal{P}}(x \le y) = \mathrm{id}$ (or 0, if $\operatorname{im}\Psi_M = 0$) for all $x \le y$ in $\mathcal{P}$. Since $\dim \operatorname{im}\Psi_M = \operatorname{rk}\Psi_M$, we have that $(\operatorname{im}\Psi_M)_{\mathcal{P}} \cong (\mathbb{I}_{\mathcal{P}})^{\operatorname{rk} M}$. Therefore, it suffices to show that $M \cong (\operatorname{im}\Psi_M)_{\mathcal{P}} \oplus N$.

Let $x \in \mathcal{P}$. The decompositions

$$\lim M \cong \operatorname{im}\Psi_M \oplus \ker\Psi_M \text{ and } \operatorname{colim} M \cong \operatorname{im}\Psi_M \oplus \operatorname{coker}\Psi_M$$

induce an isomorphism

$$\tilde{\gamma}_x \circ \tilde{\lambda}_x : \operatorname{im}\Psi_M \rightarrowtail M(x) \twoheadrightarrow \operatorname{im}\Psi_M$$

where $\tilde{\lambda}_x$ and $\tilde{\gamma}_x$ are the appropriate restrictions of λ_x and γ_x, respectively. This implies that $\operatorname{im}\Psi_M \cong \operatorname{im}\tilde{\lambda}_x \oplus \ker\tilde{\lambda}_x \cong \operatorname{im}\tilde{\lambda}_x$ since $\tilde{\lambda}_x$ is necessarily injective. Therefore, $M(x) \cong \operatorname{im}\tilde{\lambda}_x \oplus \operatorname{coker}\tilde{\lambda}_x \cong \operatorname{im}\Psi_M \oplus \operatorname{coker}\tilde{\lambda}_x$. We get the decomposition by taking $N \in \operatorname{mod}\mathcal{P}$ to be such that $N(x) = \operatorname{coker}\tilde{\lambda}_x$ and $N(x \le y) = M(x \le y)|_{N(x)}$ for all $x \le y \in \mathcal{P}$.

Since limits and colimits commute with finite direct sums, we have

$$\operatorname{im}\Psi_{(\operatorname{im}\Psi_M)_{\mathcal{P}} \oplus N} \cong \operatorname{im}(\Psi_{(\operatorname{im}\Psi_M)_{\mathcal{P}}} \oplus \Psi_N) \cong \operatorname{im}\Psi_{(\operatorname{im}\Psi_M)_{\mathcal{P}}} \oplus \operatorname{im}\Psi_N \cong \operatorname{im}\Psi_M \oplus \operatorname{im}\Psi_N$$

which implies, using the decomposition $M \cong (\operatorname{im}\Psi_M)_{\mathcal{P}} \oplus N$, that $\operatorname{im}\Psi_N = 0$. □

As a direct corollary, if M is indecomposable, then $\operatorname{rk} M \neq 0$ if and only if $M \cong \mathbb{I}_{\mathcal{P}}$.

Minimal Final Subposet

In the context of topological data analysis and more specifically, persistent homology, the generalized rank and interval modules are of particular interest. With the goal of simplifying computations, we find the minimal subposet for which the restriction along it doesn't change the generalized rank.

Definition: A functor $F : \mathcal{C} \to \mathcal{D}$ is *final* if for every $d \in \operatorname{Obj}\mathcal{D}$, the comma category d/F is (non-empty and) connected and, dually, is *initial* if F/d is connected.

Example: Let $F : \mathcal{S} \to \mathcal{P}$ be a poset homomorphism. For all $x \in \mathcal{P}$, we can compute that $\operatorname{Obj}(x/F) = \{y \in \mathcal{S} : x \le F(y)\}$ and $\operatorname{Obj}(F/x) = \{y \in \mathcal{S} : F(y) \le x\}$. If F is an order embedding, then the morphisms in x/F and F/x are given by the morphisms in $\mathcal{P}$.

Let $x \in \mathcal{P}$. We write $\uparrow x \coloneqq \{y \in \mathcal{P} : x \le y\}$ for the *upset* of x. The *downset* of x, denoted $\downarrow x$, is defined analogously.

Lemma: Suppose $F : \mathcal{S} \to \mathcal{P}$ is an order embedding and let $x \in \mathcal{P}$. The comma category x/F can be identified with $\uparrow x \cap \operatorname{im} F$ and dually, F/x can be identified with $\downarrow x \cap \operatorname{im} F$.

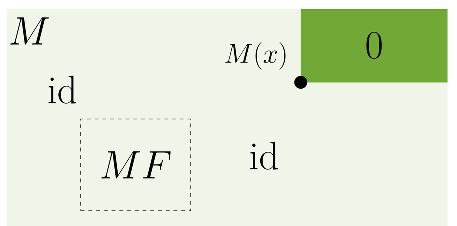
Theorem 2

Let $F : \mathcal{S} \to \mathcal{P}$ be an order embedding. Then, F is final (resp. initial) if and only if $\operatorname{colim} MF \cong \operatorname{colim} M$ (resp. $\lim MF \cong \lim M$) for all $M \in \operatorname{mod}\mathcal{P}$.

Proof: The first implication is standard, see for instance Theorem IX.3.1 in [Mac78].

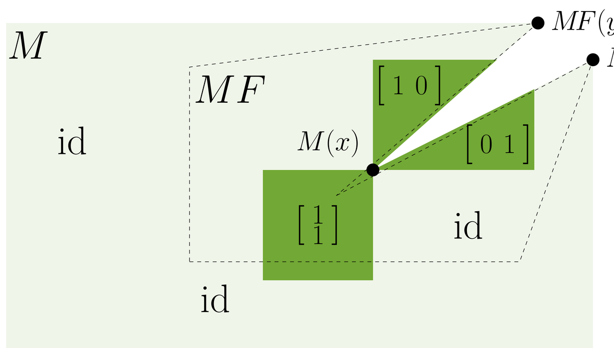
We suppose that F is not final and by the previous Lemma, we identify x/F with $\uparrow x \cap \operatorname{im} F$. There are two cases:

Suppose there exists a $x \in \mathcal{P}$ such that $\uparrow x \cap \operatorname{im} F$ is empty. We construct the functor $M : \mathcal{P} \to \operatorname{vect}_K$ with $M(x') = 0$ if $x' \in \uparrow x$ and $M(x') = K$ otherwise.



Since, for all $y \in \mathcal{S}$ we have $F(y) \notin \uparrow x$, then $MF = \mathbb{I}_{\mathcal{S}}$ and $\operatorname{colim} MF = K$. However, $\operatorname{colim} M = 0$, since, in particular, $M(x) = 0$ and $\mathcal{P}$ is connected.

Suppose now that for all $x \in \mathcal{P}$, we have that $\uparrow x \cap \operatorname{im} F$ is not empty. Suppose however that there is a $x \in \mathcal{P}$ such that $\uparrow x \cap \operatorname{im} F$ is not connected. This implies that there exists $y \neq y' \in \mathcal{S}$ where $F(y)$ and $F(y')$ are disconnected in $\uparrow x \cap \operatorname{im} F$ such that $F(y) > x < F(y')$. We choose x, y, and y' maximal with these properties. Therefore, we can construct a $\mathcal{P}$-persistence module, which we denote M, resembling:



Since $MF = \mathbb{I}_{\mathcal{S}}$ we have that $\operatorname{colim} MF = K$. On the other hand, there are maps $[1\ 0] : M(x) \to M(F(y))$ and $[0\ 1] : M(x) \to M(F(y'))$ in M so we have that $\operatorname{colim} M = 0$ and therefore $\operatorname{colim} M \not\cong \operatorname{colim} MF$. □

We are now interested in finding a minimal subposet $\mathcal{S}$ such that the embedding $F : \mathcal{S} \hookrightarrow \mathcal{P}$ is final. Define recursively the subposets $\mathcal{S}_0 \coloneqq \max\mathcal{P}$ and

$$\mathcal{S}_n \coloneqq \mathcal{S}_{n-1} \sqcup \max\{x \in \mathcal{P} : \uparrow x \cap \mathcal{S}_{n-1} \text{ is not connected}\}$$

for $n \ge 1$. We then put $\mathcal{S} \coloneqq \mathcal{S}_\infty$. Since $\mathcal{P}$ is finite, there is $\ell \ge 0$ minimal such that $\mathcal{S}_\ell = \mathcal{S}_{\ell+1} = \cdots = \mathcal{S}$. In particular, we can show that $\ell \le \#\max\mathcal{P} - 1$.

Theorem 3

The embedding $F : \mathcal{S} \hookrightarrow \mathcal{P}$ is final. Furthermore, if $F' : \mathcal{S}' \hookrightarrow \mathcal{P}$ is final, then $\mathcal{S} \subseteq \mathcal{S}'$.

Proof: We identify $\operatorname{im} F$ with $\mathcal{S}$. Suppose by contradiction that there is a $x \in \mathcal{P}$ such that $\uparrow x \cap \mathcal{S}$ is not connected. Take $x' \in \max\{x \in \mathcal{P} : \uparrow x \cap \mathcal{S} \text{ is not connected}\}$. We find that $x' \notin \mathcal{S} = \mathcal{S}_\ell$, but then $x' \in \mathcal{S}_{\ell+1} = \mathcal{S}$, which is a contradiction.

We now consider a subposet $\mathcal{S}' \subseteq \mathcal{P}$ such that the embedding $F' : \mathcal{S}' \hookrightarrow \mathcal{P}$ is final, i.e., $\uparrow x \cap \operatorname{im} F' = \uparrow x \cap \mathcal{S}'$ is connected for every $x \in \mathcal{P}$. We will show that $\mathcal{S} \subseteq \mathcal{S}'$. First, we know that $\mathcal{S}_0 = \max\mathcal{P} \subseteq \mathcal{S}'$. Suppose that $\mathcal{S}_n \subseteq \mathcal{S}'$ for a $n \ge 0$. Suppose by contradiction that $\mathcal{S}_{n+1} \nsubseteq \mathcal{S}'$ and take $y \in \mathcal{S}_{n+1} \setminus \mathcal{S}'$. Because $\mathcal{S}_n \subseteq \mathcal{S}'$, we have that $\mathcal{S}_{n+1} \setminus \mathcal{S}' \subseteq \mathcal{S}_{n+1} \setminus \mathcal{S}_n$ and therefore $y \in \max\{x \in \mathcal{P} : \uparrow x \cap \mathcal{S}_n \text{ is not connected}\}$. So, there are $s, s' \in \mathcal{S}_n$ disconnected in $\uparrow y \cap \mathcal{S}_n$ such that $s > y < s'$. But, this is impossible. Hence, we have that $\mathcal{S}_{n+1} \subseteq \mathcal{S}'$ and consequently $\mathcal{S}_\ell = \mathcal{S} \subseteq \mathcal{S}'$. □

We obtain the result for initial embedding by replacing $\max$ with $\min$ and $\uparrow x$ with $\downarrow x$.

Multiplicity

Definition: Let $M, U \in \operatorname{mod}\mathcal{P}$ where U is indecomposable. Suppose $M \cong \oplus_{i=1}^m M_i^{s_i}$ where each M_i is indecomposable, $M_i^{s_i}$ is the direct sum of s_i copies of M_i, and $M_i \not\cong M_j$ if $i \neq j$. We write

$$\operatorname{mult}(M,U) \coloneqq \begin{cases} s_i & \text{if } M_i \cong U \text{ for a } 1 \le i \le m, \\ 0 & \text{otherwise,} \end{cases}$$

and we call this number the *multiplicity* of U in M.

We are now specifically interested in $\operatorname{mult}(M, \mathbb{I}_{\mathcal{P}})$.

Theorem 4

Let $F : \mathcal{S} \to \mathcal{P}$ be an order embedding. Then, $\operatorname{mult}(M, \mathbb{I}_{\mathcal{P}}) = \operatorname{mult}(MF, \mathbb{I}_{\mathcal{S}})$ for all $M \in \operatorname{mod}\mathcal{P}$ if and only if F is final and initial.

Proof: By Theorem 1, $\operatorname{mult}(M, \mathbb{I}_{\mathcal{P}}) = \operatorname{rk} M$ and $\operatorname{mult}(MF, \mathbb{I}_{\mathcal{S}}) = \operatorname{rk} MF$. Since F is final and initial, we get from Theorem 2 the equality $\operatorname{rk} M = \operatorname{rk} MF$. □

The sufficiency part of Theorem 2 and Theorem 4 appear in [BOO24] and [DKM24]. Theorem 3 implies that the subposets considered in [DKM24] are indeed minimal, while Theorem 4 generalizes results of [AENY23], where they also study restrictions of posets.

Example: Consider the following $\mathcal{P}$-persistence module

$$M = \begin{tikzcd}[ampersand replacement=\&]
\mathbb{R}^2 \arrow[r, "\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}"] \& \mathbb{R}^2 \arrow[r, "\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}"] \& \mathbb{R}^2 \arrow[r, "\begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}"] \& \mathbb{R}^3 \arrow[r, "\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}"] \& \mathbb{R}^2 \\
\mathbb{R}^3 \arrow[u, "\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}"] \arrow[r, "\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}"] \& \mathbb{R}^3 \arrow[u, "\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}"] \arrow[r, "\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}"] \& \mathbb{R}^2 \arrow[u, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"] \arrow[r, "\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}"] \& \mathbb{R}^2 \arrow[u, "\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 1 \end{bmatrix}"] \arrow[r, "\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}"] \& \mathbb{R}^3 \arrow[u, "\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -2 \end{bmatrix}"] \\
\& \mathbb{R}^2 \arrow[u, "\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}"] \arrow[r, "\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}"] \& \mathbb{R}^2 \arrow[u, "\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}"] \arrow[r, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}"] \& \mathbb{R}^3 \arrow[u, "\begin{bmatrix} -2 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}"] \arrow[r, "\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}"] \& \mathbb{R}^2 \arrow[u, "\begin{bmatrix} 2 & 5 \\ 0 & 1 \\ 1 & 3 \end{bmatrix}"]
\end{tikzcd}$$

We want to compute $\operatorname{mult}(M, \mathbb{I}_{\mathcal{P}})$.

Let $\mathcal{S}$ be the subposet from Theorem 3 (plus its dual), i.e., the minimal subposet such that its embedding is both final and initial. Let $F : \mathcal{S} \hookrightarrow \mathcal{P}$ be the embedding functor. Then, we get the $\mathcal{S}$-persistence module

$$MF = \begin{tikzcd}[ampersand replacement=\&]
\& \& \mathbb{R}^2 \\
\mathbb{R}^3 \arrow[r, "\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}"] \& \mathbb{R}^3 \arrow[ur, "\begin{bmatrix} -1 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}"] \& \\
\& \mathbb{R}^2 \arrow[u, "\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}"] \&
\end{tikzcd}$$

We know from Theorem 4 that $\operatorname{mult}(M, \mathbb{I}_{\mathcal{P}}) = \operatorname{mult}(MF, \mathbb{I}_{\mathcal{S}})$ and we can see that $\operatorname{mult}(MF, \mathbb{I}_{\mathcal{S}}) = 1$. We can also get this from an explicit computation. By Theorem 1, we know that $\operatorname{mult}(MF, \mathbb{I}_{\mathcal{S}}) = \operatorname{rk} MF$. We can compute that $\operatorname{rk} MF = \operatorname{rk}\left(\begin{bmatrix} -1 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}\right) = 1$ which implies that $\operatorname{mult}(MF, \mathbb{I}_{\mathcal{S}}) = 1 = \operatorname{mult}(M, \mathbb{I}_{\mathcal{P}})$.

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