Minimal Initial Subcategory

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$\mathcal{C}$-Modules

Let K be a field and let $\mathcal{C}$ be a small and connected category. Write Vect_K for the category of K-vector spaces and vect_K for its full subcategory of finite-dimensional vector spaces. Denote by $\mathrm{Mod}\,\mathcal{C}$ the category of *$\mathcal{C}$-modules*, i.e., of functors $\mathcal{C} \to \mathrm{Vect}_K$ and by $\mathrm{mod}\,\mathcal{C}$ its full subcategory of functors $\mathcal{C} \to \mathrm{vect}_K$.

Example: Let G be a group. It is a category with a single object $*$ and with morphisms $\mathrm{Hom}_G(*,*) \coloneqq G$. All morphisms are isomorphisms. Then, $\mathrm{Mod}\,G$ is the category of representations of G. Indeed, a functor $F \in \mathrm{Mod}\,G$ sends an automorphism $g : * \to *$ in G to $F(g) \in \mathrm{GL}(V)$ for $V \in \mathrm{Vect}_K$.

Example: Let $\mathcal{P}$ be a connected poset. We see $\mathcal{P}$ as the category with $\mathrm{Obj}\mathcal{P} \coloneqq \mathcal{P}$ and with

$$\mathrm{Hom}_{\mathcal{P}}(x,y) \coloneqq \begin{cases} \{x \to y\} & \text{if } x \leq y, \\ \emptyset & \text{otherwise.} \end{cases}$$

Hence, if $\mathcal{S}$ and $\mathcal{P}$ are posets, then a poset homomorphism is simply a functor $F : \mathcal{S} \to \mathcal{P}$ and an order embedding ($x \leq y \iff F(x) \leq F(y)$) is a fully faithful functor. Elements of $\mathrm{Mod}\,\mathcal{P}$ are often called *$\mathcal{P}$-persistence modules.*

Example: Let Q be a connected quiver (i.e., directed graph). This is a category with the concatenation of paths as the composition and the identity morphism at a vertex is the path of no length at that vertex. The category $\mathrm{Mod}\,Q$ is equivalent to the category of representations of Q.

Definition: The *identity representation* of $\mathcal{C}$, denoted $\mathbb{I}_{\mathcal{C}} \in \mathrm{mod}\,\mathcal{C}$, is the functor which sends

$$x \xrightarrow{f} y \quad \text{to} \quad \mathbb{I}_{\mathcal{C}}(x) \xrightarrow{\mathbb{I}_{\mathcal{C}}(f)} \mathbb{I}_{\mathcal{C}}(y) \;\coloneqq\; K \xrightarrow{\mathrm{id}} K.$$

Let $M, N \in \mathrm{Mod}\,\mathcal{C}$. The *direct sum* of M and N, denoted $M \oplus N$, is the $\mathcal{C}$-module with $(M \oplus N)(x) \coloneqq M(x) \oplus N(x)$ and $(M \oplus N)(f) \coloneqq M(f) \oplus N(f)$ for all $f : x \to y \in \mathcal{C}$. If $M \cong M_1 \oplus M_2$ implies that $M_1 = 0$ or $M_2 = 0$, we say that M is *indecomposable*.

Krull-Schmidt Theorem [GR92, BCB20]

Let $M \in \mathrm{mod}\,\mathcal{C}$. Then, $M \cong \oplus_{i \in I} M_i$ where M_i is indecomposable and this decomposition is unique up to isomorphism and order of summands.

Example: The identity representation $\mathbb{I}_{\mathcal{C}}$ is indecomposable.

Remark: The direct sum gives $\mathrm{Mod}\,\mathcal{C}$ (modulo isomorphisms) the structure of an abelian group. One can define a tensor product analogously to obtain a commutative ring. The identity of that ring is the identity representation.

Generalized Rank

We write $\lim M$ and $\mathrm{colim}\, M$ for the limit and the colimit of $M \in \mathrm{Mod}\,\mathcal{C}$, respectively. Suppose $(\lambda_x)_{x \in \mathcal{C}} : \lim M \to M$ and $(\gamma_x)_{x \in \mathcal{C}} : M \to \mathrm{colim}\, M$ are the universal (co)cones. There is a canonical map $\Psi_M : \lim M \to \mathrm{colim}\, M$ given by $\gamma_x \circ \lambda_x$, for any $x \in \mathcal{C}$. The fact that $\mathcal{C}$ is connected ensures that Ψ_M is well defined, i.e., that the definition does not depend on the choice of x.

Definition [KM21]: The *generalized rank* of $M \in \mathrm{Mod}\,\mathcal{C}$ is $\mathrm{rk}\, M \coloneqq \mathrm{rk}\, \Psi_M$.

Example: Let $\mathcal{C} = \mathcal{P} = \{0,1,2\} \times \{0,1\}$ and suppose $M \in \mathrm{mod}\,\mathcal{P}$ is

$$\begin{tikzcd}
M(0,1) \arrow[r, "M_{01}^{11}"] & M(1,1) \arrow[r, "M_{11}^{21}"] & M(2,1) \\
M(0,0) \arrow[u, "M_{00}^{01}"] \arrow[r, "M_{00}^{10}"] & M(1,0) \arrow[u, "M_{10}^{11}"] \arrow[r, "M_{10}^{20}"] & M(2,0) \arrow[u, "M_{20}^{21}"]
\end{tikzcd}$$

where $M_{ij}^{kl} = M((i,j) \to (k,l))$. Then, $\lim M \cong M(0,0)$ and $\mathrm{colim}\, M \cong M(2,1)$. Pick any path $(0,0) \rightsquigarrow (2,1)$, e.g., the red one. We get $\Psi_M = M_{11}^{21} M_{10}^{11} M_{00}^{10}$. More precisely, if $x = (1,0)$, then $\lambda_x = M_{00}^{10}$ and $\gamma_x = M_{11}^{21} M_{10}^{11}$.

Theorem 1 [Kin08, CL18, BDL25]

Every $M \in \mathrm{Mod}\,\mathcal{C}$ decomposes as $M \cong (\mathbb{I}_{\mathcal{C}})^{\mathrm{rk}\, M} \oplus N$ where $\mathrm{rk}\, N = 0$.

Corollary: *If M is indecomposable, then $\mathrm{rk}\, M \neq 0$ if and only if $M \cong \mathbb{I}_{\mathcal{C}}$.*

Example: Let G be a group. Given a representation $V \in \mathrm{Mod}\,G$, we have $\lim V \cong V^G$, the fixed points of the representation, and $\mathrm{colim}\, V \cong V / \langle gv - v \mid g \in G, v \in V \rangle$. One can verify that $\mathrm{rk}\, V \leq \dim_K V^G$ and that we have equality when $\mathrm{char}\, K = 0$. By Theorem 1, this is the number of times the *trivial representation*, i.e., $\mathbb{I}_G$, arises in the decomposition of V into indecomposable representations. This is well-known from character theory, for instance when G is finite, $V \in \mathrm{mod}\,G$, and $K = \mathbb{C}$.

Minimal Initial Subposet

In the context of topological data analysis and more specifically, persistent homology, the generalized rank is of particular interest. With the goal of simplifying computations, we find the minimal subposet for which the restriction along it doesn't change the generalized rank.

Definition [Mac78, BDL25]: A functor $F : \mathcal{J} \to \mathcal{C}$ where $\mathcal{C}$ has finite hom-sets is *initial* (resp. *final*) if and only if $\lim MF \cong \lim M$ (resp. $\mathrm{colim}\, MF \cong \mathrm{colim}\, M$) for all $M \in \mathrm{mod}\,\mathcal{C}$.

Goal: Let $\mathcal{P}$ be a finite and connected poset. *We are interested in finding the minimal full subposet $\mathcal{S}$ such that the embedding $F : \mathcal{S} \hookrightarrow \mathcal{P}$ is initial.* This is closely related to finding the generalized rank:

Theorem 2 [BDL25]

Let $F : \mathcal{S} \hookrightarrow \mathcal{P}$ be an order embedding. Then, F is initial and final if and only if $\mathrm{rk}\, M = \mathrm{rk}\, MF$ for all $M \in \mathrm{mod}\,\mathcal{P}$.

Let $x \in \mathcal{P}$. We write $\downarrow x \coloneqq \{y \in \mathcal{P} : y \leq x\}$ for the *downset* of x. The *upset* of x, denoted $\uparrow x$, is defined analogously. Define recursively the subposets

$$\mathcal{S}_0 \coloneqq \min \mathcal{P} \quad \text{and} \quad \mathcal{S}_n \coloneqq \mathcal{S}_{n-1} \sqcup \min\{x \in \mathcal{P} : \downarrow x \cap \mathcal{S}_{n-1} \text{ is not connected}\}$$

for $n \geq 1$. We then put $\mathcal{S}_\infty \coloneqq \cup_{n \geq 0} \mathcal{S}_n$. Since $\mathcal{P}$ is finite, there is $\ell \geq 0$ minimal such that $\mathcal{S}_\ell = \mathcal{S}_{\ell+1} = \cdots = \mathcal{S}_\infty$. In particular, we can show that $\ell \leq \# \max \mathcal{P} - 1$.

Theorem 3 [BDL25, DL26]

The embedding $F : \mathcal{S}_\infty \hookrightarrow \mathcal{P}$ is initial, and if $G : \mathcal{Q} \hookrightarrow \mathcal{P}$ is initial, then $\mathrm{Obj}\, \mathcal{S}_\infty \subseteq \mathrm{Obj}\, \mathcal{Q}$.

Remark: Replacing $\min$ with $\max$ and $\downarrow x$ with $\uparrow x$ yields the analogous result for final embeddings.

Example: Consider the following $\mathcal{P}$-persistence module

$$M = \begin{tikzcd}[ampersand replacement=\&]
\mathbb{R}^2 \arrow[r, "{\begin{bmatrix}1\&0\\1\&1\end{bmatrix}}"] \& \mathbb{R}^2 \arrow[r, "{\begin{bmatrix}1\&-1\\0\&1\end{bmatrix}}"] \& \mathbb{R}^2 \arrow[r, "{\begin{bmatrix}-1\&1\\0\&0\\1\&0\end{bmatrix}}"] \& \mathbb{R}^3 \arrow[r, "{\begin{bmatrix}1\&0\&0\\0\&-1\&0\end{bmatrix}}"] \& \mathbb{R}^2 \\
\mathbb{R}^3 \arrow[u, "{\begin{bmatrix}1\&1\&0\\-1\&0\&-1\end{bmatrix}}"] \arrow[r, "{\begin{bmatrix}1\&0\&0\\0\&1\&0\\0\&0\&1\end{bmatrix}}"] \& \mathbb{R}^3 \arrow[u, "{\begin{bmatrix}1\&1\&0\\0\&1\&-1\end{bmatrix}}"] \arrow[r, "{\begin{bmatrix}1\&0\&1\\0\&1\&-1\end{bmatrix}}"] \& \mathbb{R}^2 \arrow[u, "{\begin{bmatrix}1\&0\\0\&1\end{bmatrix}}"] \arrow[r, "{\begin{bmatrix}-1\&1\\0\&1\end{bmatrix}}"] \& \mathbb{R}^2 \arrow[u, "{\begin{bmatrix}1\&0\\0\&0\\-1\&1\end{bmatrix}}"] \arrow[r, "{\begin{bmatrix}2\&1\\0\&1\\1\&1\end{bmatrix}}"] \& \mathbb{R}^3 \arrow[u, "{\begin{bmatrix}1\&0\&-1\\1\&1\&-2\end{bmatrix}}"] \\
\& \mathbb{R}^2 \arrow[u, "{\begin{bmatrix}1\&-1\\0\&2\\1\&1\end{bmatrix}}"] \arrow[r, "{\begin{bmatrix}1\&-1\\1\&1\end{bmatrix}}"] \& \mathbb{R}^2 \arrow[u, "{\begin{bmatrix}1\&1\\-1\&0\end{bmatrix}}"] \arrow[r, "{\begin{bmatrix}1\&0\\0\&1\\-1\&1\end{bmatrix}}"] \& \mathbb{R}^3 \arrow[u, "{\begin{bmatrix}-2\&-1\&0\\-1\&0\&0\end{bmatrix}}"] \arrow[r, "{\begin{bmatrix}0\&-1\&0\\-1\&0\&0\end{bmatrix}}"] \& \mathbb{R}^2 \arrow[u, "{\begin{bmatrix}2\&5\\0\&1\\1\&3\end{bmatrix}}"]
\end{tikzcd}.$$

Let $\mathcal{S}_\infty$ be the full subposet from Theorem 3 (plus its dual), i.e., the minimal subposet such that its embedding is both initial and final. Let $F : \mathcal{S}_\infty \hookrightarrow \mathcal{P}$ be the embedding functor. Then, we get the $\mathcal{S}_\infty$-persistence module

$$MF = \begin{tikzcd}[ampersand replacement=\&]
\& \& \mathbb{R}^2 \\
\mathbb{R}^3 \arrow[r, "{\begin{bmatrix}1\&0\&0\\0\&1\&0\\0\&0\&1\end{bmatrix}}"] \& \mathbb{R}^3 \arrow[ur, "{\begin{bmatrix}-1\&1\&-2\\0\&0\&0\end{bmatrix}}"] \& \\
\& \mathbb{R}^2 \arrow[u, "{\begin{bmatrix}1\&-1\\0\&2\\1\&1\end{bmatrix}}"] \&
\end{tikzcd}.$$

We know from Theorem 2 that $\mathrm{rk}\, M = \mathrm{rk}\, MF$. From Theorem 1, we can see that $\mathrm{rk}\, MF = 1$. We could also compute that $\lim MF \cong \mathbb{R}^2 \cong \mathrm{colim}\, MF$ and that $\mathrm{rk}\, MF = \mathrm{rk}\left(\begin{bmatrix}-1 & 1 & -2 \\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & -1 \\ 0 & 2 \\ 1 & 1\end{bmatrix}\right) = 1$. Thus, in both cases we get $\mathrm{rk}\, M = 1$.

Remark: We don't really need to take the *full* subposet with objects in $\mathcal{S}_\infty$ (plus its dual). In the above example, one could take

$$MF' = \begin{tikzcd}[ampersand replacement=\&]
\& \& \mathbb{R}^2 \\
\mathbb{R}^3 \arrow[urr, bend left, "{\begin{bmatrix}-1\&1\&-2\\0\&0\&0\end{bmatrix}}"] \arrow[r, "{\begin{bmatrix}1\&0\&0\\0\&1\&0\\0\&0\&1\end{bmatrix}}"] \& \mathbb{R}^3 \& \\
\& \mathbb{R}^2 \arrow[u, "{\begin{bmatrix}1\&-1\\0\&2\\1\&1\end{bmatrix}}"] \&
\end{tikzcd}$$

and get the right answer. This recovers the *zigzag subposet* of [DKM24].

Minimal Initial Subquiver

Once we leave the world of posets, the situation becomes much more subtle. However, it seems manageable to find minimal initial subquivers.

Example: Consider Q the quiver

$$\begin{tikzcd}
1 \arrow[r, "\alpha"] & 2 \arrow[rr, shift left, "\gamma"] \arrow[rr, shift right, "\delta"'] \arrow[dr, "\beta"'] & & 4 \arrow[r, shift right, "\zeta"'] & 5 \arrow[l, shift right, "\eta"'] \\
& & 3 \arrow[ur, "\varepsilon"'] & &
\end{tikzcd}.$$

We may form its *poset category* by setting $[x] \leq [y]$ if and only if $\mathrm{Hom}_{\mathcal{C}}(x,y) \neq \emptyset$. Then, in our case we get $\mathcal{P} \coloneqq \mathrm{Pos}\, Q \cong [1] \longrightarrow [2] \longrightarrow [3] \longrightarrow [4]$ where $[x] = \{x\}$ for $x = 1,2,3$ and $[4] = \{4,5\}$. Define an *amputation* to be a full subcategory of Q with objects a choice of $y \in [x]$ for each $[x] \in \mathrm{Pos}\, Q$. From Theorem 3 we get that $[1] = \mathcal{S}_\infty$, so $1 \in \Sigma_\infty$, the minimal initial subquiver. We further compute that

$$\Sigma_\infty = \begin{tikzcd}
1 \arrow[r, bend left, "\gamma\alpha"] \arrow[r, "\delta\alpha" description] \arrow[r, bend right, "\varepsilon\beta\alpha"'] & 4 \arrow[loop right, "\eta\zeta"]
\end{tikzcd}.$$

Example: In general a minimal initial subquiver is not unique. Simply consider $Q = \begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\beta"] \end{tikzcd}$. Then, $\mathrm{Pos}\, Q = \{1,2\}$ and the two amputations $\begin{tikzcd} 1 \arrow[loop right, "\beta\alpha"] \end{tikzcd}$ and $\begin{tikzcd} 2 \arrow[loop left, "\alpha\beta"'] \end{tikzcd}$ are minimal initial subquivers.

Minimal Projective Presentation

In many settings, the minimal projective presentation of $\mathbb{I}_{\mathcal{C}}$ seems to be related to the minimal initial subcategory. This has been observed for some posets in [DL26].

Example: Take

$$\mathcal{P} = \begin{tikzcd}
6 \arrow[r] & 7 \arrow[r] & 8 \\
3 \arrow[u] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & 4 \arrow[u] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & 5 \arrow[u] \\
& 1 \arrow[u] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & 2 \arrow[u]
\end{tikzcd} \quad \text{and} \quad Q = \begin{tikzcd}
6 \arrow[r] & 7 \arrow[r] & 8 \\
3 \arrow[u] \arrow[r] & 4 \arrow[u] \arrow[r] & 5 \arrow[u] \\
& 1 \arrow[u] \arrow[r] & 2 \arrow[u]
\end{tikzcd}$$

a poset and a quiver, respectively. One sees that

$$\mathcal{S}_\infty = \begin{tikzcd}
3 \arrow[r] & 4 \\
& 1 \arrow[u]
\end{tikzcd} \subseteq \mathcal{P} \quad \text{and} \quad \Sigma_\infty = \begin{tikzcd}
& 7 \arrow[r] & 8 \\
3 \arrow[r] & 4 \arrow[u] \arrow[r] & 5 \arrow[u] \\
& 1 \arrow[u] &
\end{tikzcd} \subseteq Q$$

are the minimal initial subcategories. The following are minimal projective presentations in their respective categories:

$$\text{in } \mathrm{mod}\,\mathcal{S}: \quad P_4 \longrightarrow P_1 \oplus P_3 \longrightarrow \mathbb{I}_{\mathcal{S}} \longrightarrow 0;$$

$$\text{in } \mathrm{mod}\,Q: \quad P_4 \oplus P_5 \oplus P_7 \oplus P_8 \longrightarrow P_1 \oplus P_3 \longrightarrow \mathbb{I}_Q \longrightarrow 0,$$

where P_x is the K-linearization of the hom-functor $\mathrm{Hom}_{\mathcal{C}}(x,-) : \mathcal{C} \to \mathrm{Set}$, for $\mathcal{C}$ either $\mathcal{P}$ or Q.

References

[BCB20] M. B. Botnan and W. Crawley-Boevey, *Decomposition of Persistence Modules*. Proceedings of the American Mathematical Society (2020).

[BDL25] T. Brüstle, J. Desrochers, and S. L., *Generalized Rank via Minimal Subposet*. (2025).

[CL18] E. W. Chambers and D. Letscher, *Persistent Homology Over Directed Acyclic Graphs*. Research in Computational Topology (2018).

[DKM24] T. K. Dey, W. Kim, and F. Mémoli, *Computing Generalized Rank Invariant for 2-Parameter Persistence Modules via Zigzag Persistence and its Applications*. Discrete & Computational Geometry (2024).

[DL26] T. K. Dey and M. Lesnick, *Limit Computation Over Posets via Minimal Initial Functors*. (2026).

[GR92] P. Gabriel and A. V. Roiter, *Algebra VIII (Chapter 3)*, Springer-Verlag (1992).

[KM21] W. Kim and F. Mémoli, *Generalized Persistence Diagrams for Persistence Modules Over Posets*. Journal of Applied and Computational Topology (2021).

[Kin08] R. Kinser, *The Rank of a Quiver Representation*. Journal of Algebra (2008).

[Mac78] S. Mac Lane, *Categories for the Working Mathematician (Chapter IX.3)*. Springer (1978).