Kernel of the Rank Invariant

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Posets

Let K be a field and let $\mathcal{P}$ be a finite and connected poset. We also denote by $\mathcal{P}$ the category with $\mathrm{Obj}\mathcal{P} = \mathcal{P}$ and with

$$\mathrm{Hom}_{\mathcal{P}}(x,y) = \begin{cases} \{x \to y\}, & \text{if } x \leq y \\ \emptyset, & \text{otherwise.} \end{cases}$$

Definition: A *$\mathcal{P}$-persistence module* is a functor $M : \mathcal{P} \to \mathrm{vect}_K$. We denote by $\mathrm{mod}\mathcal{P}$ the category of point-wise finite dimensional $\mathcal{P}$-persistence modules.

Example: Let $\mathcal{S} \subseteq \mathcal{P}$. The *indicator module* $\mathbb{L}_{\mathcal{S}}$ for $\mathcal{S}$ is the functor

$$\mathbb{L}_{\mathcal{S}}(x) = \begin{cases} K, & x \in \mathcal{S} \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \mathbb{L}_{\mathcal{S}}(x \to y) = \begin{cases} 1, & x, y \in \mathcal{S} \\ 0, & \text{otherwise} \end{cases}$$

Definition: The *split Grothendieck group* of $\mathrm{mod}\mathcal{P}$, $K_0^{\mathrm{sp}}(\mathcal{P})$, is the free abelian group generated by the isomorphism classes of objects in $\mathrm{mod}\mathcal{P}$, modulo the relation $[M] + [N] = [M \oplus N]$, for $M, N \in \mathrm{mod}\mathcal{P}$.

Quivers

Let $Q = (Q_0, Q_1, s, t)$ be a finite, connected, and acyclic quiver and K be an algebraically closed field. Let $M = (M_i, \psi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be a K-linear representation of Q. Given a path $\gamma = \alpha_1 \cdots \alpha_\ell$ in Q_1, we write the evaluation map $\psi_\gamma := \psi_{\alpha_\ell} \cdots \psi_{\alpha_1}$, the source $s(\gamma) := s(\alpha_1)$, and the target $t(\gamma) := t(\alpha_\ell)$.
Adding the condition that $\psi_\gamma = \psi_{\gamma'}$ for all paths γ and γ' such that $s(\gamma) = s(\gamma')$ and $t(\gamma) = t(\gamma')$, the category of such representations of quivers and the category of $\mathcal{P}$-persistence modules are equivalent (see for instance [HGK07]).

Gabriel's Theorem ([ASS06], VII.5.10)

There are finitely many indecomposable representations of Q if and only if Q is of type $\mathbb{A}_n$ for $n \geq 1$, $\mathbb{D}_n$ for $n \geq 4$, $\mathbb{E}_6$, $\mathbb{E}_7$, or $\mathbb{E}_8$. In this case, there are $\frac{1}{2}n(n+1)$, $n(n-1)$, 36, 63, and 120 non-isomorphic indecomposable representations, respectively.

Example: Choose an orientation of $\mathbb{A}_n$ and label the vertices by $1, \cdots, n$. For any $1 \leq i \leq j \leq n$ define the *interval* $[i,j] = \{i, i+1, \cdots, j\}$. The indecomposable representations of any orientation of $\mathbb{A}_n$ are *interval modules* of the form

$$\mathbb{L}_{[i,j]} := \begin{tikzcd} 0 \arrow[r, dash] & \cdots \arrow[r, dash] & 0 \arrow[r, dash] & K \arrow[r, dash, "1"] & \cdots \arrow[r, dash, "1"] & K \arrow[r, dash] & 0 \arrow[r, dash] & \cdots \arrow[r, dash] & 0. \end{tikzcd}$$

Example: Suppose we have the following poset of type $\mathcal{P} = \mathbb{D}_n$

$$\begin{tikzcd} & 1 \arrow[d] & & & \\ 2 \arrow[r] & 3 & 4 \arrow[l] \arrow[r, dash] & \cdots \arrow[r, dash] & n. \end{tikzcd}$$

By Gabriel's theorem, $\mathbb{D}_n$ has $n(n-1)$ indecomposable representations. The list is the following:

$$\mathbb{L}_{\{i\}},\ 1 \leq i \leq n, \quad \mathbb{L}_{\{1,2,3,\cdots,j\}},\ 3 \leq j \leq n, \quad \mathbb{L}_{\{i,\cdots,j\}},\ 2 \leq i < j \leq n, \quad \mathbb{L}_{\{1,3,\cdots,j\}},\ 3 \leq j \leq n,$$

and

$$\mathbb{M}^{\{3,\cdots,r\}}_{\{1,2,3,\cdots,j\}} := \begin{tikzcd} & K \arrow[d, "{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}"] & & & & & & & & & \\ K \arrow[r, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] & K^2 & \cdots \arrow[l, "{\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]}"'] \arrow[r, dash, "{\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]}"] & K^2 \arrow[r, dash, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"] & K \arrow[r, dash, "1"] & \cdots \arrow[r, dash, "1"] & K \arrow[r, dash] & 0 \arrow[r, dash] & \cdots \arrow[r, dash] & 0 \end{tikzcd}$$

where $4 \leq j \leq n$ and $3 \leq r < j$ are the last vertices where there is a K and a K^2, respectively.

Structure Theorem for Persistence Modules ([HGK07], 3.1.1)

Let $M \in \mathrm{mod}\mathcal{P}$. Then, $M \cong \oplus_{t=1}^n M_t$ where M_t is indecomposable and this decomposition is unique up to order. In the case where $\mathcal{P}$ is a linear order, then $M_t \cong \mathbb{L}_{[i,j]}$ for all $1 \leq t \leq n$.

References

[ASS06] I. Assem, D. Simson, and A. Skowronski, *Elements of the Representation Theory of Associative Algebras: Volume I: Techniques of Representation Theory*. Cambridge University Press (2006).

[CZ09] G. Carlsson, A. Zomorodian, *The Theory of Multidimensional Persistence*. Discrete & Computational Geometry, volume 42, pages 71–93 (2009).

[HGK07] M. Hazewinkel, N. Gubareni, and V. Kirichenko, *Algebras, Ring and Modules*. Springer, volume 2, chapters 2-3 (2007).

Rank Invariant

Definition: A *$\mathbb{Z}$-invariant*, or simply an invariant, on $\mathrm{mod}\mathcal{P}$ is a group homomorphism $\Psi : K_0^{\mathrm{sp}}(\mathcal{P}) \to \mathbb{Z}^N$. Meaning $\Psi([M \oplus N]) = \Psi([M]) + \Psi([N])$, for all $M, N \in \mathrm{mod}\mathcal{P}$.
We say that Ψ is *complete* if it is an isomorphism of groups.

Example: Denote by $\{\leq_{\mathcal{P}}\} := \{(x,y) \in \mathcal{P}^2 \mid x \leq y\}$. The *rank invariant* is the function

$$\begin{aligned} \mathbf{rk}_{\mathcal{P}} : K_0^{\mathrm{sp}}(\mathcal{P}) &\to \mathbb{Z}^{|\{\leq_{\mathcal{P}}\}|} \\ [M] &\mapsto (\mathrm{rank}(M(x) \to M(y)))_{(x,y) \in \{\leq_{\mathcal{P}}\}}. \end{aligned}$$

The following is an immediate consequence of the above structure theorem.

Theorem ([CZ09])

The rank invariant is complete for $\mathbb{A}_n$ with linear orientation.

Example: By Gabriel's theorem, there are 6 indecomposable representations a linear orientation of $\mathbb{A}_3$: $1 \longrightarrow 2 \longrightarrow 3$. They are $\mathbb{L}_{[i,j]}$ such that $1 \leq i \leq j \leq 3$. We have

$$\mathrm{rank}(\mathbb{L}_{[i,j]}(x) \to \mathbb{L}_{[i,j]}(y)) = \begin{cases} 1, & i \leq x \leq y \leq j \\ 0, & \text{otherwise} \end{cases}$$

Thus $\dim K_0^{\mathrm{sp}}(\mathcal{P}) = 6 = \dim \mathrm{im}\, \mathbf{rk}_{\mathcal{P}}$.

Incompleteness for a Non-linearly Oriented Poset of Type $\mathbb{A}_3$

In the case where $\mathcal{P} = \mathbb{A}_3$ is linearly oriented, the previous theorem implies that $\ker \mathbf{rk}_{\mathcal{P}} = 0$. This is however false in general.

Lemma

The rank invariant is not complete for $\mathbb{A}_3$: $1 \to 2 \leftarrow 3$.

Proof: We see that $\mathbf{rk}_{\mathcal{P}}([\mathbb{L}_{[1,3]} \oplus \mathbb{L}_{[2,2]}]) = \mathbf{rk}_{\mathcal{P}}([\mathbb{L}_{[1,2]} \oplus \mathbb{L}_{[2,3]}])$, but $\mathbb{L}_{[1,3]} \oplus \mathbb{L}_{[2,2]} \ncong \mathbb{L}_{[1,2]} \oplus \mathbb{L}_{[2,3]}$. $\square$

Let $M = [\mathbb{L}_{[1,3]} \oplus \mathbb{L}_{[2,2]}] - [\mathbb{L}_{[1,2]} \oplus \mathbb{L}_{[2,3]}]$. The proof above implies that $M \in \ker \mathbf{rk}_{\mathcal{P}}$. Because $\dim \mathrm{im}\, \mathbf{rk}_{\mathcal{P}} = |\{\leq_{\mathcal{P}}\}| = 5$ we have that $\dim \ker \mathbf{rk}_{\mathcal{P}} = 1$. Since $\mathrm{span}_{\mathbb{Z}}\{M\} \subseteq \ker \mathbf{rk}_{\mathcal{P}}$ and they are both of dimension 1, it follows that $\mathrm{span}_{\mathbb{Z}}\{M\} \cong \ker \mathbf{rk}_{\mathcal{P}}$.

Remark

The kernel of the rank invariant tells us exactly which $\mathcal{P}$-persistence modules cannot be distinguished by this invariant.

Motivated by the previous remark, we describe $\ker \mathbf{rk}_{\mathcal{P}}$ when $\mathcal{P}$ is a representation finite poset.

Kernel for Posets of Type $\mathbb{A}_n$

Suppose we have a quiver of type $\mathcal{P} = \mathbb{A}_n$.
Denote $\mathcal{M}(\mathcal{P}) := \{m \in \mathcal{P} \mid m \text{ is minimal or maximal}\}$.

Proposition

Denote by $G_n := \mathrm{span}_{\mathbb{Z}}\{[\mathbb{L}_{[i,j]} \oplus \mathbb{L}_{[m,m]}] - [\mathbb{L}_{[i,m]} \oplus \mathbb{L}_{[m,j]}] \mid m \in [i+1, j-1], m \in \mathcal{M}(\mathcal{P}), \text{ and } m \text{ is closest to } j\}$. Then

$$\ker \mathbf{rk}_{\mathcal{P}} \cong G_n.$$

Before proving this result, we note that if $\mathbb{A}_n$ is linearly oriented, then there are no $m \in [i+1, j-1]$ for any $i, j \in \mathcal{P}$, which implies that $\ker \mathbf{rk}_{\mathcal{P}} = 0$.

Proof: It is straightforward to see that $G_n \subseteq \ker \mathbf{rk}_{\mathcal{P}}$.
To show the isomorphism, we will proceed by induction on n. In the case $n = 3$, which is the smallest n such that $\mathbb{A}_n$ can be not linearly oriented, we know the proposition holds.
Suppose the claim true for a $n \geq 3$ and write $\mathcal{P} = \mathbb{A}_{n+1}$. There is a bijection between the indecomposable representations of $\mathbb{A}_n$ and the indecomposable representations $M = (M_i, \psi_\alpha)$ of $\mathbb{A}_{n+1}$ where we fix $M_{n+1} = 0$. Therefore, we have that our claim holds for those representations. We know by Gabriel's theorem that $n + 1 = \dim K_0^{\mathrm{sp}}(\mathbb{A}_{n+1}) - \dim K_0^{\mathrm{sp}}(\mathbb{A}_n)$, which implies that there are $n + 1$ indecomposable representations of $\mathbb{A}_{n+1}$ with $M_{n+1} = K$. Let $t \in \mathcal{M}(\mathcal{P})$ be the element closest to $n + 1$ that is different than $n + 1$. By assumption, there is a path of length at least one $t \to \cdots \to n+1$ or $t \leftarrow \cdots \leftarrow n+1$. It follows that $\dim \mathrm{im}\, \mathbf{rk}_{\mathbb{A}_{n+1}} - \dim \mathrm{im}\, \mathbf{rk}_{\mathbb{A}_n} = n - t + 2$, which implies that $\dim \ker \mathbf{rk}_{\mathbb{A}_{n+1}} - \dim \ker \mathbf{rk}_{\mathbb{A}_n} = t - 1$. We find that there are $t - 1$ vertices i such that $t \in [i+1, n+1-1]$. Since the set of all $[\mathbb{L}_{[i,j]} \oplus \mathbb{L}_{[m,m]}] - [\mathbb{L}_{[i,m]} \oplus \mathbb{L}_{[m,j]}]$ for different i is linearly independant, we conclude that $\dim G_{n+1} - \dim G_n = t - 1$ and it follows that $\ker \mathbf{rk}_{\mathbb{A}_{n+1}} \cong G_{n+1}$. $\square$

Kernel for Some Posets of Type $\mathbb{D}_n$

Suppose we have the following poset of type $\mathcal{P} = \mathbb{D}_4$

$$\begin{tikzcd} & 1 \arrow[d] & \\ 2 \arrow[r] & 3 & 4. \arrow[l] \end{tikzcd}$$

Proposition

For $n = 4$ we have

$$\ker \mathbf{rk}_{\mathcal{P}} \cong \mathrm{span}_{\mathbb{Z}} \left\{ \begin{matrix} [\mathbb{L}_{\{1,2,3\}} \oplus \mathbb{L}_{\{3\}}] - [\mathbb{L}_{\{1,3\}} \oplus \mathbb{L}_{\{2,3\}}], [\mathbb{L}_{\{1,3,4\}} \oplus \mathbb{L}_{\{3\}}] - [\mathbb{L}_{\{1,3\}} \oplus \mathbb{L}_{\{3,4\}}], \\ [\mathbb{L}_{\{2,3,4\}} \oplus \mathbb{L}_{\{3\}}] - [\mathbb{L}_{\{2,3\}} \oplus \mathbb{L}_{\{3,4\}}], [\mathbb{L}_{\{1,2,3,4\}} \oplus \mathbb{L}_{\{3\}}] - [\mathbb{M}^{\{3\}}_{\{1,2,3,4\}}], \\ [\mathbb{L}_{\{1,2,3,4\}} \oplus \mathbb{L}_{\{1,3\}}] - [\mathbb{L}_{\{1,2,3\}} \oplus \mathbb{L}_{\{1,3,4\}}] \end{matrix} \right\}$$

More generally, suppose we have the following poset of type $\mathcal{P} = \mathbb{D}_n$

$$\begin{tikzcd} & 1 \arrow[d] & & & \\ 2 \arrow[r] & 3 & 4 \arrow[l] \arrow[r, dash] & \cdots \arrow[r, dash] & n. \end{tikzcd}$$

Lemma

Let's denote

$$H_n := \mathrm{span}_{\mathbb{Z}} \left\{ \begin{aligned} a_{i,j,m} &:= [\mathbb{L}_{\{i,\dots,j\}} \oplus \mathbb{L}_{\{m\}}] - [\mathbb{L}_{\{i,\dots,m\}} \oplus \mathbb{L}_{\{m,\dots,j\}}], \\ b_j &:= [\mathbb{L}_{\{1,2,3,\dots,j\}} \oplus \mathbb{L}_{\{1,3\}}] - [\mathbb{L}_{\{1,2,3\}} \oplus \mathbb{L}_{\{1,3,\dots,j\}}], \\ c_{j,r} &:= [\mathbb{L}_{\{1,\dots,j\}} \oplus \mathbb{L}_{\{3,\dots,r\}}] - [\mathbb{M}^{\{3,\dots,r\}}_{\{1,\dots,j\}}], \\ d_{j,\ell} &:= [\mathbb{L}_{\{1,3,\dots,j\}} \oplus \mathbb{L}_{\{\ell\}}] - [\mathbb{L}_{\{1,3,\dots,\ell\}} \oplus \mathbb{L}_{\{\ell,\dots,j\}}], \\ e &:= [\mathbb{L}_{\{1,2,3\}} \oplus \mathbb{L}_{\{3\}}] - [\mathbb{L}_{\{1,3\}} \oplus \mathbb{L}_{\{2,3\}}] \end{aligned} \ \middle|\ \begin{aligned} & j \in [4, n+1], \\ & i \in [2, j-1], \\ & m, \ell \in \mathcal{M}(\mathcal{P}), \\ & m, \ell \text{ are closest to } j, \\ & m \in [i+1, j-1], \\ & \ell \in [3, j-1], \\ & r \in [3, j-1] \end{aligned} \right\}.$$

Then

$$\ker \mathbf{rk}_{\mathcal{P}} \cong H_n.$$

Proof: Direct computation of $\mathbf{rk}_{\mathcal{P}}$ shows $H_n \subseteq \ker \mathbf{rk}_{\mathcal{P}}$. It can also be shown that the elements $a_\bullet$, $b_\bullet$, $c_\bullet$, $d_\bullet$, and e are linearly independent.
To show the isomorphism, we will proceed by induction on n. In the case $n = 4$, we know the proposition holds.
Suppose the claim true for a $n \geq 4$ and write $\mathcal{P} = \mathbb{D}_{n+1}$. There is a bijection between the indecomposable representations of $\mathbb{D}_n$ and the indecomposable representations $M = (M_i, \psi_\alpha)$ of $\mathbb{D}_{n+1}$ where we fix $M_{n+1} = 0$. Therefore, we have that our claim holds for those representations. We know by Gabriel's theorem that $2n = \dim K_0^{\mathrm{sp}}(\mathbb{D}_{n+1}) - \dim K_0^{\mathrm{sp}}(\mathbb{D}_n)$, which implies that there are $2n$ indecomposable representations of $\mathbb{D}_{n+1}$ with $M_{n+1} = K$.
Let $t \in \mathcal{M}(\mathcal{P})$ be the element closest to $n + 1$ that is different than $n + 1$. By assumption, there is a path of length at least one $t \to \cdots \to n+1$ or $t \leftarrow \cdots \leftarrow n+1$. There are $n + 1 - t + 1$ pairs $(i, n+1)$ with $t \leq i \leq n+1$. It follows that $\dim \mathrm{im}\, \mathbf{rk}_{\mathbb{D}_{n+1}} - \dim \mathrm{im}\, \mathbf{rk}_{\mathbb{D}_n} = n - t + 2$, which implies that $\dim \ker \mathbf{rk}_{\mathbb{D}_{n+1}} - \dim \ker \mathbf{rk}_{\mathbb{D}_n} = n + t - 2$. Note that $e \in \ker \mathbf{rk}_{\mathbb{D}_n}$, so the new elements have the form $a_\bullet$, $b_\bullet$, $c_\bullet$, $d_\bullet$.
Fixing $j = n + 1$, we find $t - 2$ vertices $i \in [2, n+1]$ such that $t \in [i+1, n+1-1]$. This implies that there are $t - 2$ elements in H_{n+1} of the form $a_{i,n+1,t}$, which are not contained in H_n.
Again by fixing $j = n + 1$, we see there are $n - 2$ vertices r in the interval $[3, n+1-1]$. So there are $n - 2$ new elements $c_{n+1,r}$ in H_{n+1} which are not contained in H_n.
We also have a single new element b_{n+1} in H_{n+1} which is not contained in H_n. Finally, $t \in [3, n]$, $t \in \mathcal{M}(\mathcal{P})$, and t is closest to $n + 1$. So the element $d_{n+1,t}$ in H_{n+1} which is not contained in H_n.
We conclude that $\dim H_{n+1} - \dim H_n = (t-2) + (n-2) + 1 + 1 = n + t - 2$ and it follows that $\ker \mathbf{rk}_{\mathbb{D}_{n+1}} \cong H_{n+1}$. $\square$

Kernel for a Poset of Type $\mathbb{E}_6$

The technique to find $\ker \mathbf{rk}_{\mathcal{P}}$ can be adapted to any representation finite poset. Without precisely describing the kernel for a poset of type $\mathcal{P} = \mathbb{E}_6$, we will give an example of an element in $\ker \mathbf{rk}_{\mathcal{P}}$.

$$\left[\begin{tikzcd} & & K \arrow[d, "{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}"] & & \\ K \arrow[r, "1"] & K \arrow[r, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] & K^2 & K \arrow[l, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"'] & K \arrow[l, "1"'] \end{tikzcd} \oplus \begin{tikzcd} & & 0 \arrow[d] & & \\ 0 \arrow[r] & K \arrow[r, "1"] & K & K \arrow[l, "1"'] & 0 \arrow[l] \end{tikzcd} \right]$$

$$- \left[\begin{tikzcd} & & K \arrow[d, "{\left[\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right]}"] & & \\ K \arrow[r, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] & K^2 \arrow[r, "{\left[\begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix}\right]}"] & K^3 & K^2 \arrow[l, "{\left[\begin{smallmatrix}0&0\\1&0\\0&1\end{smallmatrix}\right]}"'] & K \arrow[l, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"'] \end{tikzcd} \right]$$